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ESTIMATION OF EFFECTIVE ELASTIC MODULI FOR COMPOSITES

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Abstract—The linearly elastic deformation of a composite material with matrix outer boundary is shown to be governed by the matrix deformation, inclusion deformation and deformation due to a change in the relative inclusion geometry. The latter deformation can be shown to be independent of the inclusion material property. If the effective elastic moduli of a composite are known, then we can estimate the effective elastic moduli of other composites with the same inclusion geometry and matrix material, but with different inclusion materials. This is true for any inclusion geometry and any inclusion volume fraction as long as the outer boundary of the sample consists of matrix material.

1. INTRODUCTION

The estimation of effective elastic moduli for composite materials has been extensively investigated. One is interested in how material properties of each phase and microgeometry influence the overall response of the composite materials. While the subject has been well developed, analytical expressions for effective elastic moduli for composite materials have been limited. The inclusion geometry is relatively simple such as spherical particles and fibers. The inclusion distribution is idealized and the interactions among inclusions are not well understood. An extensive review on the subject is given by Christensen (1979) and Hashin (1983).

Problems related to fluid-filled porous materials have been investigated by researchers in geotechnical engineering. They are interested in fluid flow through porous materials and interactions between a solid and a fluid. The classical theory by Biot and Willis (1957) has been widely accepted. In this theory, the effect of pore fluid pressure on the overall deformation of porous materials has been established within the linearly elastic assumption. From a point of view that is slightly different from Biot, Carroll and Katsube (1983) extended the notion of porosity to a three-dimensional case and introduced the notion of relative pore geometry change. They have shown that the overall strain due to change in relative pore geometry is determined by Terzaghi's (1923) effective stress.

Since porous materials are special cases of composite materials, the theoretical developments have similarities. Corresponding to porosity and relative pore geometry change in a porous material, we have inclusion volume fraction and relative inclusion geometry change.

Since the relative pore geometry change is shown to be governed by Terzaghi's effective stress in fluid-filled porous materials, the question arises as to what kind of role the relative inclusion geometry change plays in deformation of composite materials.

In order to answer this question, we introduce the notion of a corresponding porous material where the inclusion material is removed from the composite material. As long as the pointwise surface traction exerted from the inclusions to the matrix material is considered, the deformation of the composite material can be analysed based on the corresponding porous material. The process of cutting out the inclusion material while keeping the same tractions at the outer and inner boundaries of the matrix material is similar to the approach by Eshelby.

It is shown that the zero volume average stress in the matrix of the corresponding porous material causes zero volume average strain. This makes it possible to extend the notion of Terzaghi's effective stress to composite material as long as the hydrostatic fluid pressure inside the pores is replaced by the volume average inclusion stress tensor. The overall strain due to a change in relative inclusion geometry is determined by this extended Terzaghi's effective stress. The linearly elastic deformation of a composite material with matrix outer boundary is governed by the deformation of the matrix, inclusion and relative inclusion geometry change.

The relative inclusion geometry change measures microstructural change in composites including interactions among inclusions. In a composite material sample with matrix outer boundary, the deformation due to relative inclusion geometry change is independent of the inclusion material property. Because of this, if the effective elastic moduli of a composite sample is known, we can estimate the effective elastic moduli of other composite samples with the same inclusion geometry and matrix material but with different inclusion materials.

When the inclusion volume fraction is relatively small, then the outer boundary of the representative volume element mostly consists of matrix material. Therefore, the developed method of estimating effective elastic moduli can be used for composites with any inclusion geometry as long as the inclusion volume fraction is not large.

For the purpose of illustrating the detailed method, as in Katsube (1991), the developed method is applied to composites with spherical inclusions. The effective bulk and shear moduli for a composite material with spherical inclusions are obtained from the effective bulk and shear moduli for a porous material with spherical pores. They are identical to those obtained by Hashin's spherical composite model.

The effective bulk and shear moduli for a composite material with spherical inclusions have been applied for practical cases where the volume fraction of inclusion is reasonably large. Therefore, the developed method of estimating effective elastic moduli can possibly accommodate the range of inclusion volume fraction that typically occurs in practice.

2. PRELIMINARIES

We will use the volume average stress theorem and the volume average strain theorem as shown:

$$\overline{t_{ij}} = \frac{1}{V(R)}\int_R t_{ij}(\mathbf{x})\,\mathrm{d}v = \frac{1}{V(R)}\int_{\partial R} t_i(\mathbf{x})x_j\,\mathrm{d}a \tag{1}$$

$$\bar{e}_{ij} = \frac{1}{V(R)}\int_R e_{ij}(\mathbf{x})\,\mathrm{d}v = \frac{1}{2V(R)}\int_{\partial R} (u_i(\mathbf{x})n_j + u_j(\mathbf{x})n_i)\,\mathrm{d}a. \tag{2}$$

The stress tensor components t_{ij} and the infinitesimal strain components e_{ij} are averaged over a body occupying a region R with boundary ∂R. The volume of the region R, the components of the surface traction, the displacements and the unit outward normal on ∂R are respectively denoted by $V(R)$, t_i, u_i and n_i. The average stress theorem (1), which applies in the case of equilibrium and in the absence of body force, shows that the average stress is determined by the loading and the geometry of the deformed body, independently of the material response. The average strain theorem (2) allows us to define the strain of a region in terms of the geometry and the surface displacements. Equations (1)–(2) follow directly from the definitions of surface force and infinitesimal strain, the equation of equilibrium and the divergence theorem.

We consider the mechanical response of a sample of an anisotropic two phase composite material subject to a constant surface traction on the outer boundary B_0 as follows:

I: $t_i = t_{ij}n_j$ on B_0

where t_{ij} are constants. As long as the outer boundary is made of the matrix material, the inclusion can be of any geometry, and the volume fraction of inclusion does not have to be small.

By applying eqn (1) to the total region R, the matrix region R_m and the inclusion region R_i of the representative volume element, we respectively introduce the total average stress t_{ij}, the matrix average stress t^m_{ij} and the inclusion average stress t^i_{ij}. Similarly, by applying eqn $(2)_2$ to each region, we respectively introduce the total average strain e_{ij}, the matrix average strain e^m_{ij} and the inclusion average strain e^i_{ij}. It is important to note that since the inclusion strain is defined in terms of a surface integral through eqn $(2)_2$ it does not require the strain field inside the inclusion region. The bar to indicate volume average values is omitted for convenience.

Introducing the volume fraction of the inclusion ϕ of the representative volume element, we obtain eqns (4)–(5) from the definitions of volume average stress and strain:

$$\phi = V_i/V \tag{3}$$

$$t_{ij} = (1-\phi)t^m_{ij} + \phi t^i_{ij} \tag{4}$$

$$e_{ij} = (1-\phi)e^m_{ij} + \phi e^i_{ij}. \tag{5}$$

Equation (4) may be written as follows:

$$t_{ij} = t^m_{ij} - \frac{\phi}{1-\phi}\langle t_{ij}\rangle \tag{6}$$

where

$$\langle t_{ij}\rangle = t_{ij} - t^i_{ij}. \tag{7}$$

$\langle t_{ij}\rangle$ defined by eqn (7) may be considered as a modified Terzaghi effective stress, which has been widely used in fluid-filled porous materials.

Equation (5) may be written as follows:

$$e_{ij} = e^m_{ij} + e^*_{ij} \tag{8}$$

where

$$e^*_{ij} = \phi(e^i_{ij} - e^m_{ij}) \tag{9}$$

Equation (8) resolves the total strain of the two phase composite material into a component due to the matrix strain e^m_{ij} and one due to the change in relative inclusion geometry (or differential straining of the inclusion and the matrix) e^*_{ij}.

Within the linear assumption, the volume average response is the same as the local response. Therefore we have the following equations:

$$t^m_{ij} = M^m_{ijkl}e^m_{kl}, \quad e^m_{ij} = C^m_{ijkl}t^m_{kl} \tag{10}$$

$$t^i_{ij} = M^i_{ijkl}e^i_{kl}, \quad e^i_{ij} = C^i_{ijkl}t^i_{kl} \tag{11}$$

where M^m_{ijkl} and M^i_{ijkl} are the elastic moduli tensor components of the matrix and inclusion materials, respectively. The corresponding compliance tensor components are respectively denoted by C^{m}_{ijkl} and C^{i}_{ijkl}. The pair of compliance and moduli tensors satisfy:

$$C_{ijkl}M_{klrs} = \tfrac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}). \tag{12}$$

3. ZERO VOLUME AVERAGE STRESS AND STRAIN

Under the assumption of a sample with matrix outer boundary, the matrix is bounded by the outer boundary B_{o} and the inner boundary B_{i}.

Due to the inclusions, the stress state in the representative volume element is not uniform even though the outer boundary is subject to the constant surface traction I. Therefore, the matrix is subject to a complicated pointwise surface force on the inner boundary as follows:

$$\text{II}: \quad \begin{aligned} t_i &= t_{ik}n_k \quad \text{on } B_{\mathrm{o}} \\ t_i &= t_i(\mathbf{x}) \quad \text{on } B_{\mathrm{i}}. \end{aligned}$$

By removing the inclusion material from the composite material, we will recover a corresponding porous material where the pore geometry and porosity, respectively, are identical to the inclusion geometry and inclusion volumetric fraction. If we apply the same boundary conditions II to the corresponding porous material, the pointwise stress and strain at the corresponding points of the porous material and composite material are the same. Therefore, the overall strain of the composite material subject to the loading I is identical to that of the corresponding porous material subject to the loading II.

We now examine the deformation mechanisms of the composite material through the analysis of the corresponding porous material. Applying eqns (1) and (2) to the corresponding porous material, we can define the volume average stress and strain of the pore space. This leads to a set of equations (3)–(9) where the superscript i should be interpreted as the corresponding pore space. Focusing our attention on the corresponding porous material subject to the loading II, we decompose the loading II into parts III and IV:

$$\text{III}: \quad \begin{aligned} t_i &= t_{ik}n_k \quad \text{on } B_{\mathrm{o}} \\ t_i &= t^{\mathrm{i}}_{ik}n_k \quad \text{on } B_{\mathrm{i}} \end{aligned}$$

$$\text{IV}: \quad \begin{aligned} t_i &= 0 \quad && \text{on } B_{\mathrm{o}} \\ t_i &= t_i(\mathbf{x}) - t^{\mathrm{i}}_{ik}n_k \quad && \text{on } B_{\mathrm{i}}. \end{aligned}$$

Using eqn (1) and the divergence theorem we can show that the volume average stress of the solid matrix subject to the loading IV is zero as follows:

$$\frac{1}{V_{\mathrm{m}}}\int_{B_{\mathrm{i}}} (t_i(\mathbf{x}) - t^{\mathrm{i}}_{ik}n_k)x_j \, \mathrm{d}a = \frac{V_{\mathrm{i}}}{V_{\mathrm{m}}}\left[\frac{1}{V_{\mathrm{i}}}\int_{B_{\mathrm{i}}} t_i(\mathbf{x})x_j \, \mathrm{d}a\right] - \frac{V_{\mathrm{i}}}{V_{\mathrm{m}}} t^{\mathrm{i}}_{ij} = 0.$$

Furthermore, the loading IV causes zero volume average stress in the pore space and in the corresponding porous material. Because of the linearity, this implies zero volume average strain in the solid matrix, in the pore space and in the porous material. Therefore, as long as we limit our analysis to the volume average quantities, we can eliminate the loading IV from our discussions.

4. TERZAGHI EFFECTIVE STRESS

Within the linear assumption, the loading III for the solid matrix may be decomposed into two component loadings as follows:

$$\text{V}: \quad \begin{aligned} t_i &= t_{ij}^{\text{m}} n_j \quad \text{on } B_{\text{o}} \\ t_i &= t_{ij}^{\text{m}} n_j \quad \text{on } B_{\text{i}} \end{aligned}$$

$$\text{VI}: \quad \begin{aligned} t_i &= (t_{ij} - t_{ij}^{\text{m}}) n_j \quad \text{on } B_{\text{o}} \\ t_i &= (t_{ij}^{\text{i}} - t_{ij}^{\text{m}}) n_j \quad \text{on } B_{\text{i}}. \end{aligned}$$

The loading V causes a uniform strain in the solid matrix. The overall strain of the corresponding porous material is the same as that of the solid matrix because the loading V is achieved by filling the pores with the matrix material and applying constant surface traction $t_{ij}^{\text{m}} n_j$ on the outer boundary of the homogeneous material. Since the loadings III and V respectively cause a total strain e_{ij} and e_{ij}^{m}, from eqn (8), we conclude that the loading VI causes a total strain e_{ij}^*.

The loading VI may be further decomposed into two component loadings as follows:

$$\text{VII}: \quad \begin{aligned} t_i &= (t_{ij}^{\text{i}} - t_{ij}^{\text{m}}) n_j \quad \text{on } B_{\text{o}} \\ t_i &= (t_{ij}^{\text{i}} - t_{ij}^{\text{m}}) n_j \quad \text{on } B_{\text{i}} \end{aligned}$$

$$\text{VIII}: \quad \begin{aligned} t_i &= (1-\phi)(t_{ij}^{\text{m}} - t_{ij}^{\text{i}}) n_j \quad &&\text{on } B_{\text{o}} \\ t_i &= 0 \quad &&\text{on } B_{\text{i}}. \end{aligned}$$

The loading VII causes a total strain

$$(e_{ij})_{\text{VII}} = C_{ijkl}^{\text{m}} (t_{kl}^{\text{i}} - t_{kl}^{\text{m}}). \tag{13}$$

The loading VIII causes a total strain

$$(e_{ij})_{\text{VIII}} = (1-\phi) C_{ijkl}^{(1)} (t_{kl}^{\text{m}} - t_{kl}^{\text{i}}) \tag{14}$$

where $C_{ijkl}^{(1)}$ is the effective compliance tensor components of the corresponding porous material.

Since the total strain e_{ij}^* is the sum of the total strains due to the loadings VII and VIII as in eqn (15), eqns (4),(7),(13)–(15) lead to eqns (16) and (17):

$$e_{ij}^* = (e_{ij})_{\text{VII}} + (e_{ij})_{\text{VIII}} \tag{15}$$

$$e_{ij}^* = C_{ijkl}^* \langle t_{kl} \rangle, \quad \langle t_{ij} \rangle = M_{ijkl}^* e_{kl}^* \tag{16}$$

where

$$C_{ijkl}^* = C_{ijkl}^{(1)} - \frac{1}{1-\phi} C_{ijkl}^{\text{m}}. \tag{17}$$

In eqn (16), we have shown that the total strain due to a change in relative inclusion geometry e_{ij}^* is determined by Terzaghi effective stress. The corresponding compliance and moduli tensors are denoted by C_{ijkl}^* and M_{ijkl}^*, respectively, and they satisfy eqn (12). The effective elastic compliance tensor of the corresponding porous material $C_{ijkl}^{(1)}$ depends on the material property of the matrix and the corresponding pore geometry. Therefore eqn (17) shows that the compliance tensor C_{ijkl}^* depends on the material property of the matrix and the inclusion geometry, and it is independent of the material property of the inclusion.

5. THREE DISTINCT DEFORMATION MECHANISMS

In our point of view, the basic response equations for linearly elastic deformation of a composite material with matrix outer boundary can be summarized as follows:

$$e_{ij} = e^{m}_{ij} + e^{*}_{ij}, \quad e^{*}_{ij} = \phi(e^{m}_{ij} - e^{i}_{ij}) \tag{18}$$

$$t_{ij} = t^{m}_{ij} - \frac{\phi}{1-\phi}\langle t_{ij}\rangle, \quad \langle t_{ij}\rangle = t_{ij} - t^{i}_{ij} \tag{19}$$

$$e^{m}_{ij} = C^{m}_{ijkl}t^{m}_{kl} \tag{20}$$

$$e^{i}_{ij} = C^{i}_{ijkl}t^{i}_{kl} \tag{21}$$

$$e^{*}_{ij} = C^{*}_{ijkl}\langle t_{kl}\rangle. \tag{22}$$

The above equations are parallel to the basic response equations for linearly elastic deformation of fluid-filled porous materials. Carroll and Katsube (1983) have made clear the three deformation mechanisms of fluid-filled porous materials, i.e. the matrix response, hydrostatic fluid response and relative pore geometry change mechanism. It has also been shown that Terzaghi effective stress determines the relative pore geometry change.

In the above set of equations, we have clarified the following three points.

(1) Linearly elastic deformation of a composite sample with matrix outer boundary is governed by the three distinct deformation mechanisms, i.e. the matrix response, the inclusion response and the relative inclusion geometry change mechanism.

(2) The notion of Terzaghi effective stress can be extended to composite materials as long as we replace the hydrostatic fluid pressure by the volume average inclusion stress tensor.

(3) The relative inclusion geometry change is determined by Terzaghi effective stress.

6. EFFECTIVE ELASTIC MODULI

Based on the three distinct deformation mechanisms described in the previous section, we now obtain the effective elastic moduli for a composite material with matrix outer boundary. Inserting eqns $(10)_2$ and $(11)_2$ into eqn (5) and eliminating t^{i}_{ij} from the resulting equation and eqn (4), we obtain:

$$t^{m}_{ij} = \frac{1}{1-\phi}M^{A}_{ijkl}[e_{kl} - C^{i}_{klrs}t_{rs}] \tag{23}$$

where

$$C^{A}_{ijkl} = C^{m}_{ijkl} - C^{i}_{ijkl} \tag{24}$$

and M^{A}_{ijkl} and C^{A}_{ijkl} satisfy eqn (12).

Combining eqns (4), (7) and (16), we have:

$$e^{*}_{ij} = \frac{1-\phi}{\phi}C^{*}_{ijkl}(t^{m}_{kl} - t_{kl}). \tag{25}$$

Inserting eqns $(10)_2$ and (25) into eqn (8) and eliminating t^{m}_{ij} from the resulting equation and eqn (23), we obtain:

$$e_{ij} = \left(\frac{1}{1-\phi} C^{\mathrm{m}}_{ijkl} + \frac{1}{\phi} C^{*}_{ijkl}\right) M^{A}_{klrs} [e_{rs} - C^{\mathrm{i}}_{rspq} t_{pq}] + \frac{\phi - 1}{\phi} C^{*}_{ijkl} t_{kl}. \tag{26}$$

Rewriting eqn (26), we obtain the response equation for a composite sample as follows:

$$t_{ij} = M_{ijkl} e_{kl}, \quad e_{ij} = C_{ijkl} t_{kl} \tag{27}$$

where

$$M_{ijkl} = C^{\mathrm{i}}_{ijkl} \left[\frac{1}{2}(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk}) - \frac{1}{1-\phi}\left(C^{\mathrm{m}}_{pqrs} - \frac{1-\phi}{\phi} C^{*}_{pqrs}\right) M^{A}_{rskl}\right] \tag{28}$$

and

$$C^{B}_{ijkl} = \frac{\phi - 1}{\phi} C^{*}_{ijkl} - \frac{1}{1-\phi}\left(C^{\mathrm{m}}_{ijpq} + \frac{1-\phi}{\phi} C^{*}_{ijpq}\right) M^{A}_{pqrs} C^{\mathrm{i}}_{rskl}. \tag{29}$$

Two pairs of fourth-order tensors, M_{ijkl} and C_{ijkl}, and M^{B}_{ijkl} and C^{B}_{ijkl}, respectively satisfy eqn (10). The important point about eqn (28) with definitions (24) and (29) is that the effective elastic moduli tensor for a composite sample M_{ijkl} can be expressed in terms of C^{m}_{ijkl}, C^{i}_{ijkl} and C^{*}_{ijkl}, which describe the three distinct deformation mechanisms.

Solving the effective moduli tensor equation (28) for C^{*}_{ijkl}, we obtain:

$$C^{*}_{ijkl} = \left[\frac{1}{1-\phi} C^{\mathrm{m}}_{ijtu} M^{A}_{tumo} C^{\mathrm{i}}_{nopq} M_{pqrs} - \frac{1}{1-\phi} C^{\mathrm{m}}_{ijtu} M^{A}_{turs} + \frac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\right] C^{C}_{rskl} \tag{30}$$

where

$$M^{C}_{ijkl} = \frac{\phi - 1}{\phi} M_{ijkl} - \frac{1}{\phi} M^{A}_{ijrs} C^{\mathrm{i}}_{rspq} M_{pqkl} + \frac{1}{\phi} M^{A}_{ijkl} \tag{31}$$

and M^{C}_{ijkl} and C^{C}_{ijkl} satisfy eqn (10). The right-hand side of eqn (30) can be determined if C^{m}_{ijkl}, C^{i}_{ijkl} and C_{ijkl} are known.

Suppose that we know the effective elastic moduli tensor of a composite as well as the elastic moduli tensors of the matrix and the inclusion, and the inclusion volume fraction. Then from eqn (30), we can determine the compliance tensor of the relative inclusion geometry change C^{*}_{ijkl}. We recall from eqn (17) that C^{*}_{ijkl} is independent of the inclusion material property. Because of this, given a new composite material which has the same matrix material and inclusion geometry as the original composite but a different inclusion material, C^{*}_{ijkl} will be the same as the original composite. If we know the inclusion elastic moduli tensor of this new composite material, a set of equations (27)–(29) can be used in estimating effective elastic moduli of the new composite material.

7. ISOTROPIC CASE

When both the matrix and inclusion materials of a composite sample are isotropic and inclusions are randomly distributed and spherical, then C^{m}_{ijkl}, C^{i}_{ijkl} and C^{*}_{ijkl} become isotropic tensors. For example, we have eqns (32)–(33) corresponding to eqn (17):

$$\frac{1}{K^*} = \frac{1}{K^{(1)}} - \frac{1}{(1-\phi)K^{\mathrm{m}}} \tag{32}$$

$$\frac{1}{\mu^*} = \frac{1}{\mu^{(1)}} - \frac{1}{(1-\phi)\mu^{\mathrm{m}}} \tag{33}$$

where K^*, $K^{(1)}$, K^{m} and μ^*, $\mu^{(1)}$, μ^{m} are, respectively, the bulk and shear moduli of the relative inclusion geometry change, the corresponding porous material and the matrix.

Equation (28) reduces to:

$$K = K^{\mathrm{m}} + \frac{\phi(K^{\mathrm{i}} - K^{\mathrm{m}})(\phi K^* + (1-\phi)K^{\mathrm{m}})}{\phi K^* + (1-\phi)^2 K^{\mathrm{i}} + \phi(1-\phi)K^{\mathrm{m}}} \tag{34}$$

$$\mu = \mu^{\mathrm{m}} + \frac{\phi(\mu^{\mathrm{i}} - \mu^{\mathrm{m}})(\phi\mu^* + (1-\phi)\mu^{\mathrm{m}})}{\phi\mu^* + (1-\phi)^2\mu^{\mathrm{i}} + \phi(1-\phi)\mu^{\mathrm{m}}} \tag{35}$$

where K and μ are, respectively, the effective bulk and shear moduli of the composite.

Equation (30) reduces to:

$$K^* = \frac{(1-\phi)\{K^{\mathrm{i}}K^{\mathrm{m}} - (1-\phi)K^{\mathrm{i}}K - \phi K^{\mathrm{m}}K\}}{\phi\{K - \phi K^{\mathrm{i}} - (1-\phi)K^{\mathrm{m}}\}} \tag{36}$$

$$\mu^* = \frac{(1-\phi)\{\mu^{\mathrm{i}}\mu^{\mathrm{m}} - (1-\phi)\mu^{\mathrm{i}}\mu - \phi\mu^{\mathrm{m}}\mu\}}{\phi\{\mu - \phi\mu^{\mathrm{i}} - (1-\phi)\mu^{\mathrm{m}}\}}. \tag{37}$$

Analytical expressions of the effective bulk and shear moduli for a composite material with spherical inclusions have been obtained based on the spherical composite model (Hashin, 1962). We can recover these effective elastic moduli from the developed methods and the effective elastic moduli for a porous material with spherical pores. The effective bulk and shear moduli of a hollow sphere obtained by Mackenzie (1950) and eqns (32)–(33) lead to:

$$K^* = \frac{4(1-\phi)\mu^{\mathrm{m}}}{3\phi} \tag{38}$$

$$\mu^* = \frac{(1-\phi)\mu^{\mathrm{m}}\{9K^{\mathrm{m}} + 8\mu^{\mathrm{m}} - 5\phi(3K^{\mathrm{m}} + 4\mu^{\mathrm{m}})\}}{6\phi(K^{\mathrm{m}} + 2\mu^{\mathrm{m}})}. \tag{39}$$

Inserting eqns (38)–(39) into eqns (34)–(35), respectively, we recover the expression of the effective bulk and shear moduli for a composite with spherical inclusions:

$$K = K^{m} + \frac{\phi(K^{i} - K^{m})(3K^{m} + 4\mu^{m})}{3K^{i} + 4\mu^{m} - 3\phi(K^{i} - K^{m})} \tag{40}$$

$$\frac{\mu}{\mu^{m}} = 1 - \frac{\phi\left(1 - \dfrac{\mu^{i}}{\mu^{m}}\right)\{15K^{m} + 20\mu^{m} - 5\phi(3K^{m} + 4\mu^{m})\}}{(1-\phi)\left\{(9K^{m} + 8\mu^{m}) + \dfrac{6\mu^{i}}{\mu^{m}}(K^{m} + 2\mu^{m})\right\}}. \tag{41}$$

Using Poisson's ratio,

$$v = \frac{3K - 2\mu}{2(3K + \mu)}, \tag{42}$$

in eqn (41), we have

$$\frac{\mu}{\mu^{m}} = 1 - \frac{15\phi(1 - v^{m})\left(1 - \dfrac{\mu^{i}}{\mu^{m}}\right)}{7 - 5v^{m} + 2(4 - 5v^{m})\dfrac{\mu^{i}}{\mu^{m}}}. \tag{43}$$

Equations (40) and (43), respectively, are identical to eqns (3)–(17) and (2)–(23) in Christensen (1979).

8. DISCUSSION

If we regard a composite sample as a representative volume element, the outer boundary of an actual representative volume element consists of the matrix and inclusion materials. When the volume fraction of the inclusions is relatively small, most of the outer boundary of a representative volume element consists of the matrix material. The assumption of a representative volume element with matrix outer boundary is good only in the case where the inclusion volume fraction is not very large.

The range of inclusion volume fraction to which the developed method can be effectively applied is not clearly known. However, in composites with spherical inclusions, the validity of the current method is proven based on the spherical composite model. The spherical composite model can be effectively used for some practical cases where the volume fraction is reasonably large. Therefore, it may be reasonable to assume that the developed method can also be applied to similar situations. The key point of the developed method is that it can be applied to composites with arbitrary inclusion geometry. Theoretically the shape, size and distribution of inclusion are arbitrary as long as the inclusion volume fraction remains small. Our method can, however, possibly accommodate the range of inclusion volume fraction that typically occurs in practice based on the example shown.

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